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Master symmetries and bi-Hamiltonian structures for the relativistic Toda lattice

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Abstract. We define a bi-Hamiltonian formulation for the relativistic Toda lattice with a recursion operator on $\mathbb{R}^{2n}$. We use a theorem by W Oevel to generate higher order Poisson tensors and master symmetries for the relativistic Toda lattice. These Poisson tensors and master symmetries reduce to $\mathbb{R}^{2n-1}$.

0. Introduction

The relativistic Toda lattice was introduced by Ruijsenaars [1] and has been studied by many authors, in particular, Bruschi and Ragnisco [2, 3], Oevel *et al* [4], Suris [5] and Damianou [6]. It is a finite-dimensional completely integrable bi-Hamiltonian system. Its bi-Hamiltonian formulation and its complete integrability were proven by using various methods: Lax representation [3, 6], master symmetries [4, 6] and recursion operators [2, 4].

Fokas and Fuchssteiner [9] introduced master symmetries, also studied by Oevel and Fuchssteiner [10] and Fuchssteiner [11].

In this paper we obtain a bi-Hamiltonian formulation for the relativistic Toda Lattice (RTL) by introducing two compatible Poisson tensors on $\mathbb{R}^{2n}$ which, by a suitable projection map onto $\mathbb{R}^{2n-1}$, reduce to the two compatible Poisson tensors of the RTL. Since one of the Poisson structures introduced on $\mathbb{R}^{2n}$ is nondegenerate, we have a recursion operator and the bi-Hamiltonian structure of the RTL is, in fact, multi-Hamiltonian. Then, using a method introduced by Fernandes [7] for the nonrelativistic Toda lattice, based on a theorem due to Oevel [8], we determine master symmetries for the RTL. Our results answer a question put by Damianou in [6].

In this paper, all the manifolds, maps, vector and tensor fields are assumed to be smooth. Let us recall that a *bi-Hamiltonian manifold* is a manifold M equipped with two compatible Poisson tensors Λ_0 and Λ_1; it is denoted by $(M, \Lambda_0, \Lambda_1)$. A vector field X on M is said to be a *bi-Hamiltonian vector field* if it is Hamiltonian with respect to both Poisson structures. A *recursion operator* for $(M, \Lambda_0, \Lambda_1)$ is a vector bundle map $R : TM \to TM$ such that $\Lambda_1^\sharp = R \circ \Lambda_0^\sharp$, where $\Lambda_0^\sharp : T^*M \to TM$ and $\Lambda_1^\sharp : T^*M \to TM$ are the vector bundle maps associated with the Poisson tensors Λ_0 and Λ_1. A bi-Hamiltonian manifold for which there exists a recursion operator is called a *Poisson–Nijenhuis manifold* [12].

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1. A bi-Hamiltonian formulation for the relativistic Toda lattice

We consider $\mathbb{R}^{2n}$ with coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$ and the canonical Poisson tensor

$$\Lambda_1 = \sum_{i=1}^{n} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}. \tag{1}$$

Following Suris [5], we take $(c_1, \dots, c_{n-1}, d_1, \dots, d_n)$ as variables on $\mathbb{R}^{2n-1}$, with

$$c_i = \exp(q^i - q^{i+1} + p_i) \qquad d_i = \exp(p_i) \tag{2}$$

and we denote by $\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n-1}$ the map

$$\pi : (q^1, \dots, q^n, p_1, \dots, p_n) \mapsto (c_1, \dots, c_{n-1}, d_1, \dots, d_n). \tag{3}$$

The relativistic Toda lattice is an integrable system on $\mathbb{R}^{2n-1}$, whose equations of motion are

$$\begin{cases} \dot{c}_i = c_i(d_{i+1} - d_i + c_{i+1} - c_{i-1}) \\ \dot{d}_j = d_j(c_j - c_{j-1}) \end{cases} \tag{4}$$

where $i = 1, \dots, n-1,\ j = 1, \dots, n$, with the convention $c_0 = c_n = 0$.

The RTL is a bi-Hamiltonian system on $\mathbb{R}^{2n-1}$ for the following two compatible Poisson tensors,

$$\bar{\Lambda}_0 = \sum_{i=1}^{n-1} c_i \left(\frac{\partial}{\partial c_i} \wedge \left(\frac{\partial}{\partial d_i} - \frac{\partial}{\partial d_{i+1}} \right) + \frac{\partial}{\partial d_i} \wedge \frac{\partial}{\partial d_{i+1}} \right) \tag{5}$$

$$\bar{\Lambda}_1 = \sum_{i=1}^{n-1} c_i \frac{\partial}{\partial c_i} \wedge \left(-c_{i+1} \frac{\partial}{\partial c_{i+1}} + d_i \frac{\partial}{\partial d_i} - d_{i+1} \frac{\partial}{\partial d_{i+1}} \right) \tag{6}$$

and the following two Hamiltonians,

$$\bar{H}_0 = \sum_{i=1}^{n} (c_i + d_i) \qquad \bar{H}_1 = \sum_{i=1}^{n} (c_{i-1}(c_i + d_i) + \tfrac{1}{2}(c_i + d_i)^2). \tag{7}$$

We observe indeed that the bi-Hamiltonian vector field

$$\bar{\Lambda}_0^{\#}(d\bar{H}_1) = \bar{\Lambda}_1^{\#}(d\bar{H}_0) \tag{8}$$

is the vector field associated with the evolution equations (4) of the RTL. By convention, $c_0 = c_n = 0$ in (6) and (7).

We observe that the Poisson bracket associated with $\bar{\Lambda}_0$ is linear, while the Poisson bracket associated with $\bar{\Lambda}_1$ is quadratic.

A simple computation shows that the map $\pi : (\mathbb{R}^{2n}, \Lambda_1) \to (\mathbb{R}^{2n-1}, \bar{\Lambda}_1)$ (3) is a Poisson morphism. In other words, under that map, the canonical Poisson tensor Λ_1 on $\mathbb{R}^{2n}$ reduces to the quadratic Poisson tensor $\bar{\Lambda}_1$ on $\mathbb{R}^{2n-1}$. This behaviour differs from the behaviour of the nonrelativistic Toda lattice, for which the canonical Poisson bracket associated with Λ_1 reduces to a linear Poisson bracket on $\mathbb{R}^{2n-1}$.

Our goal is to provide a bi-Hamiltonian formulation on $\mathbb{R}^{2n}$ for the RTL. Since Λ_1 is nondegenerate, we will obtain a recursion operator for that system.

The first step is to define a Poisson tensor on $\mathbb{R}^{2n}$, compatible with the canonical Poisson tensor Λ_1, which projects onto $\mathbb{R}^{2n-1}$ under the map π (3), and has the linear Poisson tensor $\bar{\Lambda}_0$ (5) as its projection.

Proposition 1.1. Let Λ_0 be the bivector on $\mathbb{R}^{2n}$:

$$\Lambda_0 = \sum_{i=1}^{n} \exp(-p_i)\frac{\partial}{\partial q^i} \wedge \left(\frac{\partial}{\partial p_i} + \sum_{j=i+1}^{n} \frac{\partial}{\partial q^j}\right) + \sum_{i=1}^{n-1} \exp(q^i - q^{i+1} - p_{i+1}) \times \left(\left(\frac{\partial}{\partial p_i} + \frac{\partial}{\partial q^{i+1}}\right) \wedge \left(\frac{\partial}{\partial p_{i+1}} + \sum_{j=i+2}^{n} \frac{\partial}{\partial q^j}\right) - \frac{\partial}{\partial p_{i+1}} \wedge \sum_{j=i+2}^{n} \frac{\partial}{\partial q^j}\right). \qquad (9)$$

Then,

(i) $[\Lambda_0, \Lambda_0] = 0$ and $[\Lambda_0, \Lambda_1] = 0$;

(ii) $\Lambda_1^{\#}(dH_0) = \Lambda_0^{\#}(dH_1)$, with

$$H_0 = \sum_{i=1}^{n} (\exp(q^i - q^{i+1} + p_i) + \exp(p_i))$$

and

$$H_1 = \sum_{i=1}^{n} (\tfrac{1}{2}(\exp(q^i - q^{i+1} + p_i) + \exp(p_i))^2 + \exp(q^{i-1} - q^i + p_{i-1})(\exp(q^i - q^{i+1} + p_i) + \exp(p_i)))$$

where, by convention, $q^0 = -\infty$ and $q^{n+1} = +\infty$;

(iii) the map $\pi : (\mathbb{R}^{2n}, \Lambda_0) \to (\mathbb{R}^{2n-1}, \bar{\Lambda}_0)$,

$$(q^1, \ldots, q^n, p_1, \ldots, p_n) \mapsto (c_1, \ldots, c_{n-1}, d_1, \ldots, d_n)$$

is a Poisson morphism.

Proof. A simple computation leads to the required results. Relations (i) and (ii) prove the existence of a bi-Hamiltonian system on $\mathbb{R}^{2n}$, while (iii) ensures that Λ_0 reduces to $\bar{\Lambda}_0$. □

Since Λ_0 is nondegenerate, we can define a recursion operator, by setting $R = \Lambda_1^{\#} \circ (\Lambda_0^{\#})^{-1}$. We obtain

$$R = \begin{bmatrix} A & B \\ C & A^T \end{bmatrix}$$

where $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$ are $n \times n$ matrices (A^T is the transpose of A), defined as follows (with, by convention, $q^{n+1} = +\infty$):

$$\begin{cases} a_{ii} = \exp(q^i - q^{i+1} + p_i) \\ a_{i,i+1} = \exp(q^{i+1} - q^{i+2} + p_{i+1}) \\ a_{ij} = 0 & \text{if } i > j \\ a_{ij} = \exp(q^j - q^{j+1} + p_j) - \exp(q^{j-1} - q^j + p_{j-1}) & \text{if } j > i+1 \end{cases}$$

$$\begin{cases} b_{ij} = -b_{ji} \\ b_{ij} = \exp(q^j - q^{j+1} + p_j) + \exp(p_j) & \text{if } i < j \end{cases}$$

and

$$\begin{cases} c_{i+1,i} = -c_{i,i+1} = \exp(q^i - q^{i+1} + p_i) \\ c_{ij} = 0 & \text{otherwise.} \end{cases}$$

Since we have a recursion operator R, we can now define on $\mathbb{R}^{2n}$ an infinite sequence of pairwise compatible Poisson tensors Λ_k and an infinite sequence of Hamiltonians H_k, by setting $\Lambda_k = R^k \Lambda_0$, $dH_k = {}^t R(dH_{k-1})$. By the reduction theorem for bi-Hamiltonian manifolds [13], and taking account of proposition 1.1(iii), the infinite sequence (Λ_k), $k \in \mathbb{N}_0$, of higher-order Poisson tensors on $\mathbb{R}^{2n}$ reduce, under the map π, to an infinite sequence $(\bar{\Lambda}_k)$, $k \in \mathbb{N}_0$, of pairwise compatible Poisson tensors on $\mathbb{R}^{2n-1}$.

For $i = 2$, the Poisson tensor Λ_2 is given by

$$\begin{aligned}\Lambda_2 = \sum_{j=1}^{n-2} \frac{\partial}{\partial q^j} \wedge \Bigg(& \exp(q^{j+1} - q^{j+2} + p_{j+1}) \frac{\partial}{\partial p_{j+1}} \\ & + \sum_{i=j+1}^{n-1} \Bigg((-\exp(q^{i+1} - q^{i+2} + p_{i+1}) - \exp(p_{i+1})) \frac{\partial}{\partial q^i} \\ & + (\exp(q^{i+1} - q^{i+2} + p_{i+1}) - \exp(q^i - q^{i+1} + p_i)) \frac{\partial}{\partial p_{i+1}} \Bigg) \Bigg) \\ & + \sum_{i=1}^{n} \Bigg((\exp(q^i - q^{i+1} + p_i) + \exp(p_i)) \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} \\ & - \exp(q^i - q^{i+1} + p_i) \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_{i+1}} \Bigg).\end{aligned}$$

The corresponding reduced Poisson tensor $\bar{\Lambda}_2$ on $\mathbb{R}^{2n-1}$ is the Poisson tensor associated with the cubic Poisson bracket that appears in [6] and also in [4].

2. Master symmetries for the relativistic Toda lattice

Now, we want to find master symmetries for the bi-Hamiltonian system built above, in order to use the method of Fernandes [7], which is based on the following theorem.

Theorem 2.1 (Oevel). Let X_0 be a vector field on the Poisson–Nijenhuis manifold $(M, \Lambda_0, \Lambda_1)$, such that

$$\mathcal{L}(X_0)\Lambda_0 = \alpha\Lambda_0 \qquad \mathcal{L}(X_0)\Lambda_1 = \beta\Lambda_1 \qquad \text{and} \qquad X_0.H_0 = \gamma H_0 \tag{10}$$

with $\alpha, \beta, \gamma \in \mathbb{R}$. Then the vector fields $X_k = R^k X_0$ satisfy, for all $k, l \in \mathbb{N}$,

(i) $[X_k, X_l] = (\beta - \alpha)(l - k)X_{k+l}$
(ii) $[X_k, Y_l] = (\beta + \gamma + (\beta - \alpha)(l - 1))Y_{k+l}$
(iii) $\mathcal{L}(X_k)\Lambda_l = (\beta + (\beta - \alpha)(l - k - 1))\Lambda_{k+l}$
(iv) $X_k \cdot H_l = (\gamma + (\beta - \alpha)(l + k))H_{k+l}$.

In order to use this theorem, we need a vector field X_0 that satisfies conditions (10). Let us take

$$X_0 = \sum_{i=1}^{n} \frac{\partial}{\partial p_i}. \tag{11}$$

A simple computation leads to

$$\mathcal{L}(X_0)\Lambda_0 = -\Lambda_0 \qquad \mathcal{L}(X_0)\Lambda_1 = 0 \qquad X_0 \cdot H_0 = H_0.$$

So we can apply theorem 2.1 with $\alpha = -1$, $\beta = 0$ and $\gamma = 1$. Thus we have a hierarchy of master symmetries $X_k = R^k X_0$, $k \in \mathbb{N}$, that provides a way of obtaining higher-order Poisson tensors on $\mathbb{R}^{2n}$. These master symmetries satisfy the following conditions:

$$[X_k, X_l] = (l - k)X_{k+l} \tag{12}$$

$$[X_k, \Lambda_l^{\#}(dH_0)] = l\Lambda_{k+l}^{\#}(dH_0) \tag{13}$$

$$\mathcal{L}(X_k)\Lambda_l = (l - k - 1)\Lambda_{k+l} \tag{14}$$

$$X_k \cdot H_l = (1 + l + k)H_{k+l}. \tag{15}$$

Proposition 2.1. The master symmetries $X_k = R^k X_0$, $k \in \mathbb{N}$, are projectable vector fields by the map $\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n-1}$, $(q^i, p_i) \mapsto (c_i, d_i)$. We denote by $\bar{X}_k$ the projected vector fields.

Proof. The fibres of π are integral curves of the vector field

$$Z = \sum_{i=1}^{n} \frac{\partial}{\partial q^i}.$$

Since $Z = -\sum_{i=1}^{n} \Lambda_1^{\#}(dp_i)$, this implies $[\Lambda_1, Z] = 0$.

Further we obtain, by computation, $[\Lambda_0, Z] = 0$ and therefore $[R, Z] = 0$. Also, $[X_0, Z] = 0$ and we deduce

$$[X_k, Z] = [R^k X_0, Z] = 0.$$

For any $f \in C^{\infty}(\mathbb{R}^{2n}, \mathbb{R})$, we have

$$\begin{aligned}[X_k, fZ] &= (X_k.f)Z + f[X_k, Z] \\ &= (X_k.f)Z\end{aligned}$$

which proves that X_k is a projectable vector field. □

Now, if we take the reduced vector fields $\bar{X}_k$, the reduced Poisson tensors $\bar{\Lambda}_k$, the reduced Hamiltonian vector fields $\bar{Y}_k = \bar{\Lambda}_k^{\#}(d\bar{H}_0)$ and the reduced Hamiltonians $\bar{H}_k$ then, taking account of relations (12), (13), (14) and (15), we deduce the following relations:

$$[\bar{X}_k, \bar{X}_l] = (l-k)\bar{X}_{k+l} \tag{16}$$

$$[\bar{X}_k, \bar{Y}_l] = l\bar{Y}_{k+l} \tag{17}$$

$$\mathcal{L}(\bar{X}_k)\bar{\Lambda}_l = (l-k-1)\bar{\Lambda}_{k+l} \tag{18}$$

$$\bar{X}_k \cdot \bar{H}_l = (1+l+k)\bar{H}_{k+l}. \tag{19}$$

Some of these relations already appeared in [6], although our vector field $\bar{X}_1$ differs from the corresponding one in [6]—let us denote it by $\dot{X}_1$, by a bi-Hamiltonian vector field. In fact, we compute

$$\begin{aligned}X_1 = \sum_{i=1}^{n} \Bigg(&(1-i)(\exp(q^i - q^{i+1} + p_i) + \exp(p_i)) \\ &+ \sum_{j=i+1}^{n} (\exp(q^j - q^{j+1} + p_j) + \exp(p_j)) \Bigg) \frac{\partial}{\partial q^i} \\ &+ \sum_{i=1}^{n} (\exp(p_i) + \mathrm{i}\exp(q^i - q^{i+1} + p_i) + (2-i)\exp(q^{i-1} - q^i + p_{i-1})) \frac{\partial}{\partial p_i}\end{aligned}$$

where, by convention, $q^0 = -\infty$ and $q^{n+1} = +\infty$, and X_1 projects onto

$$\begin{aligned}\bar{X}_1 = \sum_{i=1}^{n} \Bigg(&((1+i)c_i(c_{i+1} + d_{i+1}) + (2-i)c_i(c_{i-1} + d_i) + c_i^2) \frac{\partial}{\partial c_i} \\ &+ (ic_i d_i + (2-i)c_{i-1}d_i + d_i^2) \frac{\partial}{\partial d_i} \Bigg)\end{aligned}$$

where, by convention, $c_0 = c_n = 0$. Comparing $\bar{X}_1$ with $\dot{X}_1$, we obtain

$$\dot{X}_1 - \bar{X}_1 = \bar{\Lambda}_1^{\#}(d\bar{H}_0) = \bar{\Lambda}_0^{\#}(d\bar{H}_1).$$

Since the difference is the bi-Hamiltonian vector field $\bar{\Lambda}_1^{\#}(d\bar{H}_0)$, our higher-order Poisson tensors $\bar{\Lambda}_k$ coincide with those of [6].

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